Hume = Small Hume

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In *Frege's Conception of Numbers as Objects* (1983), Crispin Wright proposed replacing Frege's inconsistent Basic Law V,

Basic Law V: $Ext(X) = Ext(Y) \leftrightarrow \forall x(X(x) \leftrightarrow Y(x))$

with the following abstraction axiom for cardinal numbers:

Hume: $\#X = \#Y \leftrightarrow X \approx Y$

Here, X and Y are understood as classes, or concepts, or higher-order entities which are not first-order objects, and the formula $X \approx Y$ expresses, in the dyadic second-order language, the equinumerosity of X and Y. The first-order entity $\#X$ is the cardinal number associated with X. This axiom is sometimes called *Hume's Principle* and the formal system obtained by adding (Hume) to the second-order logic is called *Frege Arithmetic*. Discussing Wright's work, Burgess (1984), Hodes (1984) and Hazen (1985) noted that Frege Arithmetic has a (countable) model, and it was subsequently verified (by Boolos 1986/7, 1987, and others) that second-order arithmetic Z_2 is interpretable within Frege Arithmetic.

In 1989 Boolos proposed a weakened modification of Basic Law V,

New V: $Ext^*(X) = Ext^*(Y) \leftrightarrow [S(X) \vee S(Y) \rightarrow \forall x(X(x) \leftrightarrow Y(x))]$

where the formula $S(X)$ expresses that X is 'small': that is, X is not equinumerous with the universe V of first-order entities. That is, $S(X)$ is short for $\neg(X \approx \mathrm{V})$. The idea here is to implement the basic 'limitation of size' intuition – sets are (or correspond to) classes which aren't too big. Boolos referred to such first-order entities as $Ext^*(X)$ as 'subtensions'. As Boolos went on to show, the set theory based on (New V) is consistent, but surprisingly weak. For example, it does not prove the axiom of infinity or the power set axiom.

For entertainment, consider the analogous modification of (Hume):

Small Hume: $\#X = \#Y \leftrightarrow [S(X) \vee S(Y) \rightarrow X \approx Y]$

At first sight, one would guess that (Small Hume) is weaker than (Hume). However, we have:

$\vdash_{\mathrm{SOL}}$ (Hume) $\leftrightarrow$ (Small Hume)

For the $\Rightarrow$ direction, assume (Hume) and suppose (Small Hume) fails. Then there are classes X, Y such that *either* (i) $\#X = \#Y$, and either X or Y is small, but $\neg(X \approx Y)$; *or* (ii) $\#X \neq \#Y$ and if either X or Y is small, then $X \approx Y$. In case (i) we have $\#X \neq \#Y$, from (Hume), a contradiction. In case

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(ii), $\neg(X \approx Y)$, from (Hume). Thus, neither X nor Y is small. Hence, both are equinumerous with V and thus to each other, so $X \approx Y$. Contradiction.

For the $\Leftarrow$ direction, assume (Small Hume) and suppose (Hume) fails. Then there are classes X, Y such that *either* (i) $\#X = \#Y$ and $\neg(X \approx Y)$ *or* (ii) $\#X \neq \#Y$ and $X \approx Y$. In case (i), from (Small Hume), if either X or Y is small, then $X \approx Y$. Hence, neither X nor Y is small. Again, both are equinumerous with V and so $X \approx Y$. Contradiction. In case (ii), (Small Hume) entails that either X or Y is small, and $\neg(X \approx Y)$. Contradiction.[1]

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References

Boolos, G. 1986/7. Saving Frege from contradiction. *Proceedings of the Aristotelian Society* 87: 137–51. Reprinted in Boolos 1998.

Boolos, G. 1987. The consistency of Frege's *Foundations of Arithmetic*. In *On Being and Saying: Essays for Richard Cartwright*, ed. J. J. Thompson. Cambridge, Mass.: MIT Press, 1987. Reprinted in Boolos 1998.

Boolos, G. 1989. Iteration again. *Philosophical Topics* 42: 5–21. Reprinted in Boolos 1998.

Boolos, G. 1998. *Logic, Logic and Logic*. Cambridge, Mass.: Harvard University Press.

Burgess, J. 1984. Review of Wright 1983. *Philosophical Review* 93: 638–40.

Frege, G. 1893. *Grundgesetze der Arithmetik*. Vol.1. Jena.

Hazen, A. 1985. Review of Wright 1983. *Australasian Journal of Philosophy* 63: 251–54.

Hodes, H. 1984. Logicism and the ontological commitments of arithmetic. *Journal of Philosophy* 81: 123–49.

Wright, C. 1983. *Frege's Conception of Numbers as Objects*. Aberdeen: Aberdeen University Press.

[1] Richard Heck pointed out to me that this simple result hinges on the fact that in such theories there exists a 'rogue object' (viz. #V), which here corresponds to the size of the universe V.